

# FACTORIZATION OF THE IDENTITY THROUGH OPERATORS WITH LARGE DIAGONAL

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**ABSTRACT.** Given a Banach space  $X$  with an unconditional basis, we consider the following question: does the identity on  $X$  factor through every bounded operator on  $X$  with large diagonal relative to the unconditional basis? We show that on Gowers' space with its unconditional basis there exists an operator for which the answer to the question is negative. By contrast, for any operator on the mixed-norm Hardy spaces  $H^p(H^q)$ , where  $1 \leq p, q < \infty$ , with the bi-parameter Haar system, this problem always has a positive solution. The one-parameter  $H^p$  spaces were treated first by Andrew [1] in 1979.

## 1. INTRODUCTION

Let  $X$  be a Banach space. A *basis* for  $X$  will always mean a Schauder basis. We denote by  $I_X$  the identity operator on  $X$ , and write  $\langle \cdot, \cdot \rangle$  for the duality bracket between  $X$  and its dual space  $X^*$ . By an *operator* on  $X$ , we understand a bounded and linear mapping from  $X$  into itself.

Suppose that  $X$  has a normalized basis  $(b_n)_{n \in \mathbb{N}}$ , and let  $b_n^* \in X^*$  be the  $n^{\text{th}}$  coordinate functional. For an operator  $T$  on  $X$ , we say that:

- ▷  $T$  has *large diagonal* if  $\inf_{n \in \mathbb{N}} |\langle Tb_n, b_n^* \rangle| > 0$ ;
- ▷  $T$  is *diagonal* if  $\langle Tb_m, b_n^* \rangle = 0$  whenever  $m, n \in \mathbb{N}$  are distinct;
- ▷ *the identity operator on  $X$  factors through  $T$*  if there are operators  $R$  and  $S$  on  $X$  such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{I_X} & X \\ R \downarrow & & \uparrow S \\ X & \xrightarrow{T} & X \end{array}$$

is commutative.

Suppose that the basis  $(b_n)_{n \in \mathbb{N}}$  for  $X$  is unconditional. Then the diagonal operators on  $X$  correspond precisely to the elements of  $\ell_\infty(\mathbb{N})$ , and so for each operator  $T$  on  $X$  with large diagonal, there is a diagonal operator  $S$  on  $X$  such that  $\langle STb_n, b_n^* \rangle = 1$  for each  $n \in \mathbb{N}$ . This observation naturally leads to the following question.

**Question 1.1.** *Can the identity operator on  $X$  be factored through each operator on  $X$  with large diagonal?*

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In classical Banach spaces such as  $\ell^p$  with the unit vector basis and  $L^p$  with the Haar basis, the answer to this question is known to be positive. These are the theorems of Pelczynski [11] and Andrew [1], respectively; see also [7, Chapter 9].

The aim of the present paper is to establish the following two results.

- ▷ There exists a Banach space with an unconditional basis for which the answer to Question 1.1 is negative.
- ▷ Question 1.1 has a positive answer for the mixed-norm Hardy space  $H^p(H^q)$ , where  $1 \leq p, q < \infty$ , with the bi-parameter Haar system as its unconditional basis. This conclusion can be viewed as a bi-parameter extension of Andrew's theorem [1] on the perturbability of the one-parameter Haar system in  $L^p$ .

The precise statements of these results, together with their proofs, are given in Sections 2 and 3, respectively.

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## 2. THE ANSWER TO QUESTION 1.1 IS NOT ALWAYS POSITIVE

The aim of this section is to establish the following result, which answers Question 1.1 in the negative.

**Theorem 2.1.** *There is an operator  $T$  on a Banach space  $X$  with an unconditional basis such that  $T$  has large diagonal, but the identity operator on  $X$  does not factor through  $T$ .*

The proof of Theorem 2.1 relies on two ingredients. The first of these is Fredholm theory, which we shall now recall the relevant parts of.

Given an operator  $T$  on a Banach space  $X$ , we set

$$\alpha(T) = \dim \ker T \in \mathbb{N}_0 \cup \{\infty\} \quad \text{and} \quad \beta(T) = \dim(X/T(X)) \in \mathbb{N}_0 \cup \{\infty\},$$

and we say that:

- ▷  $T$  is an *upper semi-Fredholm operator* if  $\alpha(T) < \infty$  and  $T$  has closed range;
- ▷  $T$  is a *Fredholm operator* if  $\alpha(T) < \infty$  and  $\beta(T) < \infty$ .

Note that the condition  $\beta(T) < \infty$  implies that  $T$  has closed range (see, *e.g.*, [4, Corollary 3.2.5]), so that each Fredholm operator is automatically upper semi-Fredholm. For an upper semi-Fredholm operator  $T$ , we define its *index* by

$$i(T) = \alpha(T) - \beta(T) \in \mathbb{Z} \cup \{-\infty\}.$$

The main property of the class of upper semi-Fredholm operators that we shall require is that it is stable under strictly singular perturbations in the following precise sense. Let  $T$  be an upper semi-Fredholm operator on a Banach space  $X$ , and suppose that  $S$  is an operator on  $X$  which is *strictly singular* in the sense that, for each  $\varepsilon > 0$ , every infinite-dimensional subspace of  $X$  contains a unit vector  $x$  such that  $\|Sx\| \leq \varepsilon$ . Then  $T + S$  is an upper semi-Fredholm operator, and

$$i(T + S) = i(T).$$

A proof of this result can be found in [9, Proposition 2.c.10].

We shall require the following piece of notation in the proof of our next lemma. For an element  $x$  of a Banach space  $X$  and a functional  $f \in X^*$ , we write  $x \otimes f$  for the rank-one operator on  $X$  defined by

$$(x \otimes f)y = \langle y, f \rangle x \quad (y \in X).$$

**Lemma 2.2.** *Let  $T$  be a diagonal upper semi-Fredholm operator on a Banach space with a basis. Then  $\beta(T) = \alpha(T)$ , so that  $T$  is a Fredholm operator with index 0.*

*Proof.* Let  $X$  be the Banach space on which  $T$  acts, and let  $(b_n)_{n \in \mathbb{N}}$  be the basis for  $X$  with respect to which  $T$  is diagonal. Set  $N = \{n \in \mathbb{N} : Tb_n = 0\}$ . Since  $T$  is diagonal, we have  $\ker T = \overline{\text{span}}\{b_n : n \in N\}$ , and so the set  $N$  is finite, with  $\alpha(T)$  elements. Consequently, we can define a projection of  $X$  onto  $\ker T$  by  $P_N = \sum_{n \in N} b_n \otimes b_n^*$ . The fact that  $\ker P_N = \overline{\text{span}}\{b_n : n \in \mathbb{N} \setminus N\}$  implies that  $T(X) \subseteq \ker P_N$ . Conversely, for each  $n \in \mathbb{N} \setminus N$ , we have  $b_n = T(\langle Tb_n, b_n^* \rangle^{-1} b_n)$ , so we conclude that  $\ker P_N \subseteq T(X)$  because  $T$  has closed range. Hence

$$\beta(T) = \dim P_N(X) = \alpha(T) < \infty,$$

and the result follows.  $\square$

The other main ingredient in the proof of Theorem 2.1 is the Banach space  $X_G$  which Gowers [5] created to solve Banach's hyperplane problem. This Banach space has subsequently been investigated in more detail by Gowers and Maurey [6, Section (5.1)]. Its main properties are as follows.

**Theorem 2.3** (Gowers [5]; Gowers and Maurey [6]). *There is a Banach space  $X_G$  with an unconditional basis such that each operator on  $X_G$  is the sum of a diagonal operator and a strictly singular operator.*

**Corollary 2.4.** *Each upper semi-Fredholm operator on the Banach space  $X_G$  is a Fredholm operator of index 0.*

*Proof.* Let  $T$  be an upper semi-Fredholm operator on  $X_G$ . By Theorem 2.3, we can find a diagonal operator  $D$  and a strictly singular operator  $S$  on  $X_G$  such that  $T = D + S$ . The stability of the class of upper semi-Fredholm operators under strictly singular perturbations that we stated above implies that  $D$  is an upper semi-Fredholm operator with the same index as  $T$ , and hence the conclusion follows from Lemma 2.2.  $\square$

*Proof of Theorem 2.1.* Let  $X = X_G$  be the Banach space from Theorem 2.3, and let  $(b_n)_{n \in \mathbb{N}}$  be the unconditional basis for  $X_G$  with respect to which each operator on  $X_G$  is the sum of a diagonal operator and a strictly singular operator. We may suppose that  $(b_n)_{n \in \mathbb{N}}$  is normalized. Set

$$T = I_{X_G} + b_1 \otimes b_2^* + b_2 \otimes b_1^*.$$

Then  $T$  has large diagonal because  $\langle Tb_n, b_n^* \rangle = 1$  for each  $n \in \mathbb{N}$ .

Assume towards a contradiction that  $I_{X_G} = STR$  for some operators  $R$  and  $S$  on  $X_G$ . Then  $R$  is injective, and its range is complemented (because  $RST$  is a projection onto it), and it is thus closed, so that  $R$  is an upper semi-Fredholm operator with  $\alpha(R) = 0$ . This implies that  $R$  is a Fredholm operator of index 0 by Corollary 2.4, and hence  $R$  is invertible. Since  $ST$  is a left inverse of  $R$ , the uniqueness of the inverse shows that  $R^{-1} = ST$ , but this contradicts that the operator  $T$  is not injective (because  $T(b_1 - b_2) = 0$ ).  $\square$

As we have seen in the proof of Theorem 2.1, the identity operator need not factor through a Fredholm operator. If, however, we allow ourselves sums of two operators, then we can always factor the identity operator, as the following result shows.

**Proposition 2.5.** *Let  $T$  be a Fredholm operator on an infinite-dimensional Banach space  $X$ . Then there are operators  $R_1, R_2, S_1$ , and  $S_2$  on  $X$  such that*

$$I_X = S_1 T R_1 + S_2 T R_2.$$

*Proof.* Let  $P = \sum_{j=1}^n x_j \otimes f_j$  be a projection of  $X$  onto the kernel of  $T$ , where  $n \in \mathbb{N}$ ,  $x_1, \dots, x_n \in X$  and  $f_1, \dots, f_n \in X^*$ , and let  $Q$  be a projection of  $X$  onto the range of  $T$ . Since this range is infinite-dimensional, we can find  $y_1, \dots, y_n \in X$  and  $g_1, \dots, g_n \in X^*$  such that  $\langle Ty_j, g_k \rangle = \delta_{j,k}$  (the Kronecker delta) for each  $j, k \in \{1, \dots, n\}$ . The restriction  $\tilde{T}: x \mapsto Tx, \ker P \rightarrow T(X)$ , is invertible, so we may define an operator on  $X$  by  $S_1 = J\tilde{T}^{-1}Q$ , where  $J: \ker P \rightarrow X$  is the inclusion. Set

$$R_1 = I_X - P, \quad R_2 = \sum_{j=1}^n y_j \otimes f_j, \quad \text{and} \quad S_2 = \sum_{k=1}^n x_k \otimes g_k.$$

Then, for each  $z \in X$ , we have

$$\begin{aligned} (S_1 T R_1 + S_2 T R_2)z &= J\tilde{T}^{-1}QT(z - Pz) + \sum_{j,k=1}^n \langle Ty_j, g_k \rangle \langle z, f_j \rangle x_k \\ &= (z - Pz) + Pz = z, \end{aligned}$$

from which the conclusion follows.  $\square$

### 3. THE IDENTITY FACTORS THROUGH OPERATORS WITH LARGE DIAGONAL ON MIXED-NORM HARDY SPACES

Let  $\mathcal{D}$  denote the collection of dyadic subintervals of the unit interval  $[0, 1]$ , and let  $h_I$  be the  $L^\infty$ -normalized Haar function supported on  $I \in \mathcal{D}$ ; that is, for  $I = [a, b]$  and  $c = (a + b)/2$ , we have  $h_I(x) = 1$  if  $a \leq x < c$ ,  $h_I(x) = -1$  if  $c < x \leq b$ , and  $h_I(x) = 0$  otherwise. Moreover, let  $\mathcal{R} = \{I \times J : I, J \in \mathcal{D}\}$  be the collection of dyadic rectangles contained in the unit square, and set

$$h_{I \times J}(x, y) = h_I(x)h_J(y) \quad (I \times J \in \mathcal{R}, x, y \in [0, 1]).$$

For  $1 \leq p, q < \infty$ , the *mixed-norm Hardy space*  $H^p(H^q)$  is the completion of

$$\text{span}\{h_{I \times J} : I \times J \in \mathcal{R}\}$$

under the square function norm

$$\|f\|_{H^p(H^q)} = \left( \int_0^1 \left( \int_0^1 \left( \sum_{I \times J} |a_{I \times J}|^2 h_{I \times J}^2(x, y) \right)^{q/2} dy \right)^{p/q} dx \right)^{1/p}, \quad (3.1)$$

where  $f = \sum_{I \times J} a_{I \times J} h_{I \times J}$ . Then  $(h_{I \times J})_{I \times J \in \mathcal{R}}$  is an unconditional basis of  $H^p(H^q)$ , called the *bi-parameter Haar system*. We note that:

- ▷ Let  $1 < p, q < \infty$  and  $\frac{1}{p} + \frac{1}{p'} = 1$ ,  $\frac{1}{q} + \frac{1}{q'} = 1$ . Then the dual of  $H^p(H^q)$  naturally identifies with  $H^{p'}(H^{q'})$ . Similarly, for the limiting cases we have  $H^1(H^q)^* = BMO(H^{q'})$ ,  $H^p(H^1)^* = H^{q'}(BMO)$  and  $H^1(H^1)^* = BMO(BMO)$ . See Maurey [10].
- ▷ Since the bi-parameter Haar system is an unconditional basis, we do not need to specify an ordering of its index set  $\mathcal{R}$ .
- ▷ This basis is  $L^\infty$  normalized and not normalized in  $H^p(H^q)$ ; we have  $\|h_{I \times J}\|_{H^p(H^q)} = |I|^{1/p} |J|^{1/q}$ .
- ▷ Hence, an operator  $T : H^p(H^q) \rightarrow H^p(H^q)$  has large diagonal with respect to the  $L^\infty$ -normalized Haar system  $(h_{I \times J})_{I \times J \in \mathcal{R}}$  if for some  $\delta > 0$  we have that  $|\langle Th_{I \times J}, h_{I \times J} \rangle| \geq \delta$  for all  $I \times J \in \mathcal{R}$ .

We can now state the main result of this section precisely.

**Theorem 3.1.** *Let  $1 \leq p, q < \infty$ , and  $\delta > 0$ , and let  $T : H^p(H^q) \rightarrow H^p(H^q)$  be an operator satisfying*

$$|\langle Th_{I \times J}, h_{I \times J} \rangle| \geq \delta |I \times J| \quad \text{for all } I \times J \in \mathcal{R}.$$

Then the identity operator on  $H^p(H^q)$  factors through  $T$ , that is, there are operators  $R$  and  $S$  such that the diagram

$$\begin{array}{ccc} H^p(H^q) & \xrightarrow{I_{H^p(H^q)}} & H^p(H^q) \\ R \downarrow & & \uparrow S \\ H^p(H^q) & \xrightarrow{T} & H^p(H^q) \end{array} \quad (3.2)$$

is commutative. Moreover, there is a universal constant  $C > 0$ , dependent only on  $p$  and  $q$ , such that the operators  $R$  and  $S$  can be chosen with  $\|R\| \|S\| \leq C/\delta$ .

Given a pairwise disjoint family  $\{\mathcal{B}_{I \times J} : I \times J \in \mathcal{R}\}$ , where each set  $\mathcal{B}_{I \times J}$  is a collection of disjoint dyadic rectangles, we define

$$b_{I \times J}(x, y) = \sum_{K \times L \in \mathcal{B}_{I \times J}} h_{K \times L}(x, y) \quad (x, y \in [0, 1]), \quad (3.3)$$

and we call  $\{b_{I \times J}\}$  the *block basis generated by  $\{\mathcal{B}_{I \times J}\}$* . The system  $\{b_{I \times J}\}$  is equivalent to the Haar system  $\{h_{I \times J}\}$  if the linear map

$$B: \sum a_{I \times J} h_{I \times J} \mapsto \sum a_{I \times J} b_{I \times J}$$

is bounded with respect to the norm (3.1), and its extension to  $H^p(H^q)$  is an isomorphism onto its range. In this case, we set  $C = \|B\| \|B^{-1}\|$ , and we say that  $\{b_{I \times J}\}$  is *C-equivalent* to  $\{h_{I \times J}\}$ .

### 3.1. Capon's local product condition.

Capon [3] discovered a condition on a collection of the form  $\{\mathcal{B}_{I \times J}\}$  which ensures that the block basis  $\{b_{I \times J}\}$  given by (3.3) is equivalent to the Haar system  $\{h_{I \times J}\}$  in  $H^p(H^q)$ . We shall now describe this condition in detail.

First of all, Capon considers collections of dyadic rectangles of the form:

$$\mathcal{B}_{I \times J} = \{K \times L : K \in \mathcal{X}_I, L \in \mathcal{Y}_J(K)\},$$

where  $I, J \in \mathcal{D}$  and  $\mathcal{X}_I, \mathcal{Y}_J(K) \subset \mathcal{D}$ . For  $I, J, K \in \mathcal{D}$ , we set

$$\mathcal{X} = \bigcup_{I \in \mathcal{D}} \mathcal{X}_I, \quad X_I = \bigcup_{K \in \mathcal{X}_I} K, \quad \text{and} \quad Y_J(K) = \bigcup_{L \in \mathcal{Y}_J(K)} L.$$

Moreover, for  $J \in \mathcal{D}$  and  $x \in [0, 1]$ , we define

$$Y_{J,x} = \bigcap \{Y_J(K) : K \in \mathcal{X}, K \ni x\} \quad \text{and} \quad Y_J^x = \bigcup \{Y_J(K) : K \in \mathcal{X}, K \ni x\}. \quad (3.4)$$

We say that the collection  $\{\mathcal{B}_{I \times J} : I \times J \in \mathcal{R}\}$  satisfies Capon's *local product condition* if there are constants  $C_X, C_Y > 0$  such that the following ten conditions (X1)–(X5) and (Y1)–(Y5) are satisfied.

- (X1) For each  $I \in \mathcal{D}$ ,  $\mathcal{X}_I$  is a collection of pairwise disjoint dyadic intervals, and  $\mathcal{X}_{I_0} \cap \mathcal{X}_{I_1} = \emptyset$  whenever  $I_0, I_1 \in \mathcal{D}$  are distinct.
- (X2) Let  $I_0, I \in \mathcal{D}$ ,  $K_0 \in \mathcal{X}_{I_0}$ , and  $K \in \mathcal{X}_I$ . Then  $K_0 \subset K$  if and only if  $I_0 \subset I$ .
- (X3) For all  $I, I_0, I_1 \in \mathcal{D}$  with  $I_0 \cap I_1 = \emptyset$  and  $I = I_0 \cup I_1$ , we have

$$X_{I_0} \cap X_{I_1} = \emptyset \quad \text{and} \quad X_{I_0} \cup X_{I_1} \subset X_I.$$

- (X4) For each  $I \in \mathcal{D}$ , we have

$$C_X^{-1}|I| \leq |X_I| \leq C_X|I|.$$

- (X5) For all  $I_0, I \in \mathcal{D}$  with  $I_0 \subset I$  and  $K \in \mathcal{X}_I$ , we have

$$\frac{|K \cap X_{I_0}|}{|K|} \geq \frac{|X_{I_0}|}{C_X|X_I|}.$$

- (Y1) For  $J \in \mathcal{D}$  and  $K \in \mathcal{X}$ ,  $\mathcal{Y}_J(K)$  is a collection of pairwise dyadic intervals, and  $\mathcal{Y}_{J_0}(K) \cap \mathcal{Y}_{J_1}(K) = \emptyset$  whenever  $J_0, J_1 \in \mathcal{D}$  with  $J_0 \neq J_1$  and  $K \in \mathcal{X}$ .
- (Y2) Let  $J_0, J \in \mathcal{D}$ ,  $K \in \mathcal{X}$ ,  $L_0 \in \mathcal{Y}_{J_0}(K)$ , and  $L \in \mathcal{Y}_J(K)$ . Then  $L_0 \subset L$  if and only if  $J_0 \subset J$ .
- (Y3) For all  $J, J_0, J_1 \in \mathcal{D}$  with  $J_0 \cap J_1 = \emptyset$ ,  $J = J_0 \cup J_1$  and  $x \in [0, 1]$ , we have

$$\begin{aligned} Y_{J_0, x} \cap Y_{J_1, x} &= \emptyset, & Y_{J_0, x} \cup Y_{J_1, x} &\subset Y_{J, x}, \\ Y_{J_0}^x \cap Y_{J_1}^x &= \emptyset, & Y_{J_0}^x \cup Y_{J_1}^x &\subset Y_J^x. \end{aligned}$$

- (Y4) For each  $J \in \mathcal{D}$  and  $x \in [0, 1]$ , we have

$$C_Y^{-1}|J| \leq |Y_{J, x}| \quad \text{and} \quad |Y_J^x| \leq C_Y|J|.$$

- (Y5) For all  $I_0, I, J_0, I \in \mathcal{D}$  with  $I_0 \subset I$  and  $J_0 \subset J$ ,  $K \in \mathcal{X}_I$ ,  $K_0 \in \mathcal{X}_{I_0}$ , and  $L \in \mathcal{Y}_J(K)$ , we have

$$\frac{|L \cap Y_{J_0}(K_0)|}{|L|} \geq \frac{|Y_{J_0}(K_0)|}{C_Y|Y_J(K)|}.$$

We remark that Capon's local product structure immediately implies the identity

$$b_{I \times J}(x, y) = \sum_{K \times L \in \mathcal{B}_{I \times J}} h_{K \times L}(x, y) = \sum_{K \in \mathcal{X}_I} h_K(x) \sum_{L \in \mathcal{Y}_J(K)} h_L(y).$$

### 3.2. Block bases and projections in $H^p(H^q)$ .

Capon proved in [3] that if the conditions (X1)–(X5) and (Y1)–(Y5) are satisfied, then  $\{b_{I \times J} : I \times J \in \mathcal{R}\}$  is equivalent to  $\{h_{I \times J} : I \times J \in \mathcal{R}\}$  in  $H^p(H^q)$ ,  $1 \leq p, q < \infty$ , and thereby that the orthogonal projection  $Q : H^p(H^q) \rightarrow H^p(H^q)$  given by

$$Qf = \sum_{I \times J} \frac{\langle f, b_{I \times J} \rangle}{\|b_{I \times J}\|_2^2} b_{I \times J} \quad (3.5)$$

is bounded on  $H^p(H^q)$ ,  $1 < p, q < \infty$ . To see this, note that

$$\begin{aligned} \|Q\| &= \sup_{\|f\| \leq 1} \|Qf\| \\ &\simeq \sup_{\|f\| \leq 1} \left\| \sum_R \frac{\langle f, b_R \rangle}{\|b_R\|_2^2} b_R \right\| \\ &= \sup_{\|f\|, \|g\| \leq 1} \left| \sum_R \frac{\langle h_R, g \rangle}{\|b_R\|_2^2} \langle f, b_R \rangle \right| \\ &\simeq \sup_{\|f\|, \|g\| \leq 1} \left| \left\langle \sum_R \frac{\langle h_R, g \rangle}{\|h_R\|_2^2} b_R, f \right\rangle \right| \\ &= \sup_{\|g\| \leq 1} \left\| \sum_R \frac{\langle h_R, g \rangle}{\|h_R\|_2^2} b_R \right\| \\ &\simeq \sup_{\|g\| \leq 1} \left\| \sum_R \frac{\langle h_R, g \rangle}{\|h_R\|_2^2} h_R \right\| \\ &= 1 \end{aligned}$$

As far as the boundedness of the orthogonal projection is concerned, Capon's proof does not cover the cases  $p = 1$  or  $q = 1$ . The following theorem extends Capon's result to parameter range  $1 \leq p, q < \infty$ .

**Theorem 3.2.** *Let  $1 \leq p, q < \infty$ . Assume that the pairwise disjoint collection of pairwise disjoint dyadic rectangles  $\{\mathcal{B}_{I \times J} : I \times J \in \mathcal{R}\}$  satisfies Capon's local product condition (X1)–(X5) and (Y1)–(Y5) with constants  $C_X$  and  $C_Y$ . Then there is a constant  $C > 0$ , which depends only on  $C_X$  and  $C_Y$ , such that the following two conditions hold.*

- (i) The block basis  $\{b_{I \times J} : I \times J \in \mathcal{R}\}$  is  $C$ -equivalent to the bi-parameter Haar basis in  $H^p(H^q)$ .
- (ii) Equation (3.5) defines a projection  $Q$  on  $H^p(H^q)$ , and

$$\|Qf\|_{H^p(H^q)} \leq C\|f\|_{H^p(H^q)} \quad (f \in H^p(H^q)).$$

*Proof.* At a certain point, we will distinguish the following three cases.

- (1)  $1 \leq p < \infty, q = 1$ ,
- (2)  $p = 1, 1 \leq q \leq 2$ ,
- (3)  $p = 1, 2 \leq q < \infty$ .

By duality and interpolating between these spaces, see [2], we obtain the boundedness of  $Q$  on  $H^p(H^q)$  for all  $1 \leq p, q < \infty$ .

Let  $N_0 \in \mathbb{N}$  and denote by  $\mathcal{R}_{N_0}$  the collection of indices  $\{I_0 \times J_0 \in \mathcal{R} : |I_0| = 2^{-N_0}, |J_0| = 2^{-N_0}\}$ , and let  $\mathcal{B}_{N_0}$  denote the collection of building blocks given by

$$\mathcal{B}_{N_0} = \{K_0 \times L_0 \in \mathcal{B}_{I_0 \times J_0} : I_0 \times J_0 \in \mathcal{R}_{N_0}\}.$$

We may assume that  $f$  is a finite linear combination of bi-parameter Haar functions supported only on the building blocks above  $\mathcal{B}_{N_0}$ , i.e.

$$f = \sum_{K_0 \times L_0 \in \mathcal{B}_{N_0}} \sum_{K \times L \supset K_0 \times L_0} a_{K \times L} h_{K \times L}.$$

First, we will estimate  $\|Qf\|_{H^p(H^q)}^p$ . To this end, note that

$$\|Qf\|_{H^p(H^q)}^p = \int_0^1 \left( \int_0^1 \left( \sum_{I \times J} \frac{\langle f, b_{I \times J} \rangle^2}{\|b_{I \times J}\|_2^4} \sum_{K \in \mathcal{R}_I} \mathbb{1}_K(x) \mathbb{1}_{Y_J(K)}(y) \right)^{q/2} dy \right)^{p/q} dx.$$

By the definition of  $Y_J^x$ , see (3.4) we have  $\mathbb{1}_{Y_J(K)}(y) \leq \mathbb{1}_{Y_J^x}(y)$ , for all  $x \in K \in \mathcal{R}$ , thus by discretizing the inner integral with  $\{Y_{J_0}^x : |J_0| = 2^{-N_0}\}$  and using (Y3) and (Y4) we get

$$\|Qf\|_{H^p(H^q)}^p \lesssim \int_0^1 \left( \sum_{|J_0|=2^{-N_0}} |J_0| \left( \sum_I \sum_{J \supset J_0} \frac{\langle f, b_{I \times J} \rangle^2}{\|b_{I \times J}\|_2^4} \mathbb{1}_{X_I}(x) \right)^{q/2} \right)^{p/q} dx.$$

Now we discretize the remaining integral with  $\{X_{I_0} : |I_0| = 2^{-N_0}\}$  and use (X3) and (X4) to obtain the following upper estimate for  $\|Qf\|_{H^p(H^q)}^p$ :

$$\sum_{I_0=2^{-N_0}} |I_0| \left( \sum_{|J_0|=2^{-N_0}} |J_0| \left( \sum_{I \times J \supset I_0 \times J_0} \left( \sum_{K \times L \in \mathcal{B}_{I \times J}} \frac{|a_{K \times L}| |K| |L|}{|I| |J|} \right)^2 \right)^{q/2} \right)^{p/q}. \quad (3.6)$$

Note that by (X4), (3.4) and (Y4)  $|B_{I \times J}| \gtrsim |I| |J|$ .

Second, since the collection  $\mathcal{B}_{I \times J}$  consists of disjoint dyadic rectangles  $K \times L$ , we have by (X2) and (Y2)

$$\mathbb{S}^2(f) = \sum_{I_0 \times J_0 \in \mathcal{B}_{N_0}} \sum_{I \times J \supset I_0 \times J_0} \left( \sum_{K \times L \in \mathcal{B}_{I \times J}} |a_{K \times L}| \mathbb{1}_{K \times L} \mathbb{1}_{B_{I_0 \times J_0}} \right)^2.$$

For each  $x$  there exists at most one  $K_0 \in \mathcal{R}_{I_0}$  such that  $K_0 \ni x$ . If it exists, we denote this  $K_0$  by  $K_0(x)$ , and otherwise, we define  $K_0(x) = \emptyset$ . Furthermore, we abbreviate  $Y_{J_0}(x) = Y_{J_0}(K_0(x))$ . Now we write  $\mathbb{1}_{B_{I_0 \times J_0}}(x, y) = \mathbb{1}_{K_0(x)}(x) \mathbb{1}_{Y_{J_0}(x)}(y)$  and put

$$F_{J_0}(x) = \int_{Y_{J_0}(x)} \left( \sum_{|I_0|=2^{-N_0}} \sum_{I \times J \supset I_0 \times J_0} \left( \sum_{K \times L \in \mathcal{B}_{I \times J}} |a_{K \times L}| \mathbb{1}_{K \times L}(x, y) \mathbb{1}_{K_0(x)}(x) \mathbb{1}_{Y_{J_0}(x)}(y) \right)^2 \right)^{q/2} \frac{dy}{|Y_{J_0}(x)|}. \quad (3.7)$$

For each  $x$  we discretize the inner integral with  $Y_{J_0}(x)$  and obtain by (Y3) and (Y4)

$$\|f\|_{H^p(H^q)}^p = \int_0^1 \left( \sum_{|J_0|=2^{-N_0}} |J_0| F_{J_0}(x) \right)^{p/q} dx. \quad (3.8)$$

It is now time to divide the proof into separate cases.

**Case 1:  $1 \leq p < \infty$ ,  $q = 1$ .** Here, we estimate  $F_{J_0}(x)$  by Minkowski's inequality for integrals and  $\ell^2$  norm, i.e.

$$F_{J_0}(x) \geq \left( \sum_{|I_0|=2^{-N_0}} \sum_{I \times J \supset I_0 \times J_0} \left( \sum_{K \times L \in \mathcal{B}_{I \times J}} |a_{K \times L}| \mathbb{1}_K(x) \mathbb{1}_{K_0(x)}(x) \frac{|L \cap Y_{J_0}(x)|}{|Y_{J_0}(x)|} \right)^2 \right)^{q/2}.$$

Since we have  $\mathbb{1}_{K_0(x)}(x) = \mathbb{1}_{X_{I_0}}(x)$ ,  $Y_{J_0,x} \subset Y_{J_0}(x) \subset Y_{J_0}^x$ , (Y5) and (Y4), we obtain

$$F_{J_0}(x) \gtrsim \left( \sum_{|I_0|=2^{-N_0}} \sum_{I \times J \supset I_0 \times J_0} \left( \sum_{K \times L \in \mathcal{B}_{I \times J}} \frac{|a_{K \times L}| |L|}{|J|} \mathbb{1}_K(x) \mathbb{1}_{X_{I_0}}(x) \right)^2 \right)^{1/2}. \quad (3.9)$$

Next, we combine the estimates (3.9) and (3.8), discretize the outer integral with  $\{X_{I_0} : |I_0| = 2^{-N_0}\}$  and observe that due to (X3) we have

$$\|f\|_{H^p(H^1)}^p \gtrsim \sum_{|I_0|=2^{-N_0}} \int_{X_{I_0}} \left( \sum_{|J_0|=2^{-N_0}} |J_0| \left( \sum_{I \times J \supset I_0 \times J_0} \left( \sum_{K \times L \in \mathcal{B}_{I \times J}} \frac{|a_{K \times L}| |L|}{|J|} \mathbb{1}_K(x) \mathbb{1}_{X_{I_0}}(x) \right)^2 \right)^{1/2} F_{J_0}(x) \right)^p \frac{dx}{|X_{I_0}|}.$$

Applying Jensen's inequality and then Minkowski's inequality for  $\ell^2$  norm as well as using (X3), (X5) and (X4), we obtain (3.6) for all  $1 \leq p < \infty$ ,  $q = 1$ .

**Case 2:  $p = 1$ ,  $1 \leq q \leq 2$ .** Similar as above, we estimate  $F_{J_0(x)}$  by Minkowski's inequality for integrals and  $\ell^{2/q}$  norm, and then Jensen's inequality for the convex function  $t \mapsto t^q$  to obtain

$$F_{J_0}(x) \gtrsim \left( \sum_{|I_0|=2^{-N_0}} \sum_{I \times J \supset I_0 \times J_0} \left( \sum_{K \times L \in \mathcal{B}_{I \times J}} \frac{|a_{K \times L}| |L|}{|J|} \mathbb{1}_K(x) \mathbb{1}_{X_{I_0}}(x) \right)^2 \right)^{q/2}. \quad (3.10)$$

Again, we also used that  $\mathbb{1}_{K_0(x)}(x) = \mathbb{1}_{X_{I_0}}(x)$ ,  $Y_{J_0,x} \subset Y_{J_0}(x) \subset Y_{J_0}^x$  and (Y5), (Y4). We combine the estimates (3.10) and (3.8), discretize the outer integral with  $\{X_{I_0} : |I_0| = 2^{-N_0}\}$  and apply Minkowski's inequality, this time for  $\ell^q$  and  $\ell^2$  norm. Using (X3), (X5) and (X4), we obtain (3.6) for all  $p = 1$ ,  $1 \leq q \leq 2$ .

**Case 3:  $p = 1$ ,  $2 \leq q < \infty$ .** Here, we estimate  $F_{J_0}(x)$  by using Jensen's inequality on the convex functions  $t \mapsto t^{q/2}$  and  $t \mapsto t^2$ . Since  $\mathbb{1}_{K_0(x)}(x) = \mathbb{1}_{X_{I_0}}(x)$ ,  $Y_{J_0,x} \subset Y_{J_0}(x) \subset Y_{J_0}^x$  we have (Y5) and (Y4), we obtain estimate (3.10). After plugging (3.10) into (3.8), discretizing the outer integral with  $\{X_{I_0} : |I_0| = 2^{-N_0}\}$  and applying Minkowski's inequality for  $\ell^q$  and  $\ell^2$  norm, we use (X3), (X5) and (X4), to obtain (3.6) for all  $p = 1$ ,  $2 \leq q < \infty$ .

□



0		4		16	
				15	
		3		14	
				13	
1	2	6		20	24
		8		19	23
				18	22
		5		17	21
9	10	11	12	26	28
				30	32
				36	40
				35	39
				44	48
				43	47
				34	38
				42	46
				33	37
				41	45
				25	27
				29	31

FIGURE 1. The first 49 rectangles.

*Remark 3.3.* The above proof shows as well that the orthogonal projection onto more general block basis such as

$$b_{I \times J}(x, y) = \sum_{K \times L \in \mathcal{B}_{I \times J}} \alpha_{K \times L} h_{K \times L}(x, y)$$

is bounded on  $H^p(H^q)$ ,  $1 \leq p, q < \infty$  provided that  $\sup_{I \times J} |\alpha_{I \times J}| < \infty$ .

### 3.3. Proof of Theorem 3.1.

Our proof of Theorem 3.1 is by induction. Hence, we first introduce a suitable linear ordering  $\triangleleft$  on the collection of dyadic rectangles  $\mathcal{R}$ . We choose  $\triangleleft$  in such a way that on  $\mathcal{R}$  the bijective index function  $\mathcal{O}_{\triangleleft} : \mathcal{R} \rightarrow \mathbb{N}_0$ , which is defined by

$$\mathcal{O}_{\triangleleft}(R_0) < \mathcal{O}_{\triangleleft}(R_1) \Leftrightarrow R_0 \triangleleft R_1, \quad R_0, R_1 \in \mathcal{R},$$

has the properties (3.11) and (3.12): it links the geometry of a dyadic rectangle to its position

$$(2^k - 1)^2 \leq \mathcal{O}_{\triangleleft}(I \times J) < (2^{k+1} - 1)^2, \quad \text{whenever } \min(|I|, |J|) = 2^{-k}, \quad (3.11)$$

and it respects the position of dyadic predecessors

$$\tilde{I} \times J \triangleleft I \times J, \text{ for } I \neq [0, 1] \quad \text{and} \quad I \times \tilde{J} \triangleleft I \times J, \text{ for } J \neq [0, 1]. \quad (3.12)$$

For a picture of such an index function  $\mathcal{O}_{\triangleleft}$  see Figure 1. For more details see [8].

*Proof.* Given  $I \times J \in \mathcal{R}$  we write

$$Th_{I \times J} = \alpha_{I \times J} h_{I \times J} + r_{I \times J}, \quad (3.13a)$$

where

$$\alpha_{I \times J} = \frac{\langle Th_{I \times J}, h_{I \times J} \rangle}{|I \times J|} \quad \text{and} \quad r_{I \times J} = \sum_{E \times F \neq I \times J} \frac{\langle Th_{I \times J}, h_{E \times F} \rangle}{|E \times F|} h_{E \times F}. \quad (3.13b)$$

We note the estimates

$$\delta \leq |\alpha_{I \times J}| \leq \|T\| \quad \text{and} \quad \|r_{I \times J}\|_{H^p(H^q)} \leq 2\|T\| |I|^{1/p} |J|^{1/q}. \quad (3.14)$$

**Inductive construction of  $b_{I \times J}^{(\varepsilon)}$ .**

Given a small parameter  $\eta > 0$ , we will now inductively define a block basis  $\{b_{I \times J}^{(\varepsilon)} : I \times J \in \mathcal{R}\}$  for a suitable choice of signs  $\varepsilon$ . From here on, we will regularly identify a rectangle  $I \times J \in \mathcal{R}$  with its index  $\mathcal{O}_\Delta(I \times J)$ . To begin the induction we set

$$\mathcal{B}_{[0,1] \times [0,1]} = \{[0,1] \times [0,1]\} \quad \text{and} \quad b_{[0,1] \times [0,1]}^{(\varepsilon)} = h_{[0,1] \times [0,1]}. \quad (3.15)$$

Now we assume that  $\mathcal{B}_j$  has already been constructed for all  $1 \leq j < i$ , a suitable choice of signs  $\varepsilon_{K \times L} \in \{-1, +1\}$  has been made for all  $K \times L \in \bigcup_{j < i} \mathcal{B}_j$  and the block basis elements  $b_j^{(\varepsilon)}$  are given by

$$b_j^{(\varepsilon)} = \sum_{K \times L \in \mathcal{B}_j} \varepsilon_{K \times L} h_{K \times L}. \quad (3.16)$$

We will now construct a collection  $\mathcal{B}_i$  and choose signs  $\varepsilon_{K \times L} \in \{-1, +1\}$  for all  $K \times L \in \mathcal{B}_i$  such that

$$b_i^{(\varepsilon)} = \sum_{K \times L \in \mathcal{B}_i} \varepsilon_{K \times L} h_{K \times L}, \quad (3.17a)$$

$$\sum_{j=1}^{i-1} |\langle T b_j^{(\varepsilon)}, b_i^{(\varepsilon)} \rangle| + |\langle b_i^{(\varepsilon)}, T^* b_j^{(\varepsilon)} \rangle| \leq \eta 4^{-i} \|b_i^{(\varepsilon)}\|_2^2, \quad (3.17b)$$

$$\langle T b_i^{(\varepsilon)}, b_i^{(\varepsilon)} \rangle \geq (\delta - \eta) \|b_i^{(\varepsilon)}\|_2^2. \quad (3.17c)$$

Let  $I \times J \in \mathcal{R}$  such that  $\mathcal{O}_\Delta(I \times J) = i$ . We distinguish between the cases

$$J = [0, 1], \quad J \neq [0, 1], I = [0, 1] \quad \text{and} \quad J \neq [0, 1], I \neq [0, 1].$$

**Case 1:  $J = [0, 1]$ .** Here, we know that  $I \neq [0, 1]$ . Let  $\tilde{I}$  be the dyadic predecessor of  $I$ , then  $\mathcal{B}_{\tilde{I} \times [0,1]}$  has already been defined. We note that by our previous choices we have that

$$K \times L \in \mathcal{B}_{\tilde{I} \times [0,1]} \quad \text{implies} \quad L = [0, 1].$$

For a dyadic interval  $K_0$  we denote its left half by  $K_0^\ell$  and its right half by  $K_0^r$ . We define the sets

$$B_{\tilde{I} \times [0,1]}^\ell = \bigcup_{K_0 \times [0,1] \in \mathcal{B}_{\tilde{I} \times [0,1]}} K_0^\ell \times [0, 1]$$

and

$$B_{\tilde{I} \times [0,1]}^r = \bigcup_{K_0 \times [0,1] \in \mathcal{B}_{\tilde{I} \times [0,1]}} K_0^r \times [0, 1].$$

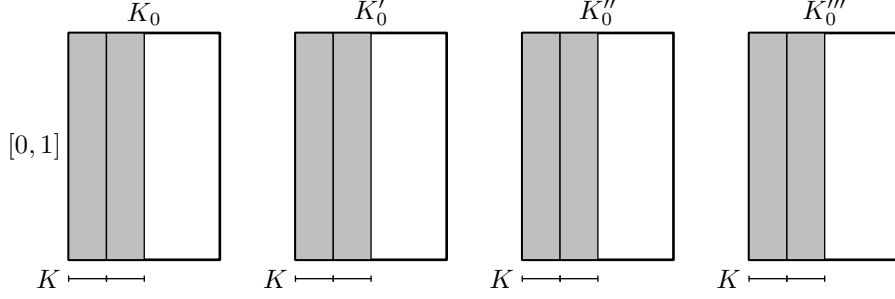
Now we perform a Gamlen-Gaudet step in the  $x$ -component. If  $I$  is the *left half* of  $\tilde{I}$  we put

$$\mathcal{F}_m = \{K \times [0, 1] \in \mathcal{R} : |K| = 2^{-m}, K \times [0, 1] \subset B_{\tilde{I} \times [0,1]}^\ell\}. \quad (3.18a)$$

If  $I$  is the *right half* of  $\tilde{I}$  we define

$$\mathcal{F}_m = \{K \times [0, 1] \in \mathcal{R} : |K| = 2^{-m}, K \times [0, 1] \subset B_{\tilde{I} \times [0,1]}^r\}, \quad (3.18b)$$

see Figure 2.


 FIGURE 2. The above figure depicts an instance of  $\mathcal{F}_m$  in Case 1.

**Case 2:**  $J \neq [0, 1]$ . In this case we know that  $\mathcal{B}_{I \times [0, 1]}$  has already been constructed. We define the set of  $x$ -frequencies simply by putting

$$\mathcal{X}_{I \times J} = \{K_0 : K_0 \times [0, 1] \in \mathcal{B}_{I \times [0, 1]}\}.$$

**Case 2.a:**  $J \neq [0, 1], I = [0, 1]$ . We remark that  $\mathcal{B}_{[0, 1] \times \tilde{J}}$  has already been constructed. We note that by our previous choices we have that  $\mathcal{X}_{I \times J} = \{[0, 1]\}$ , so

$$K \times L \in \mathcal{B}_{[0, 1] \times \tilde{J}} \text{ implies } K = [0, 1].$$

Define the sets

$$B_{[0, 1] \times \tilde{J}}^\ell = \bigcup_{[0, 1] \times L_0 \in \mathcal{B}_{[0, 1] \times \tilde{J}}} [0, 1] \times L_0^\ell$$

and

$$B_{[0, 1] \times \tilde{J}}^r = \bigcup_{[0, 1] \times L_0 \in \mathcal{B}_{[0, 1] \times \tilde{J}}} [0, 1] \times L_0^r.$$

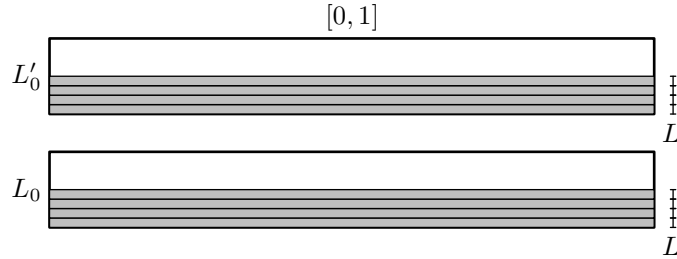
If  $J$  is the *left half* of  $\tilde{J}$  we put

$$\mathcal{F}_m = \{[0, 1] \times L \in \mathcal{R} : |L| = 2^{-m}, [0, 1] \times L \subset B_{[0, 1] \times \tilde{J}}^\ell\}. \quad (3.19a)$$

If  $I$  is the *right half* of  $\tilde{I}$  then

$$\mathcal{F}_m = \{[0, 1] \times L \in \mathcal{R} : |L| = 2^{-m}, [0, 1] \times L \subset B_{[0, 1] \times \tilde{J}}^r\}, \quad (3.19b)$$

see Figure 3.


 FIGURE 3. The above figure depicts an instance of  $\mathcal{F}_m$  in Case 2.a.

**Case 2.b:**  $J \neq [0, 1], I \neq [0, 1]$ . Note that in this case  $\mathcal{B}_{\tilde{I} \times J}$  has already been defined. Let

$$B_{\tilde{I} \times J} = \bigcup_{K_1 \times L_1 \in \mathcal{B}_{\tilde{I} \times J}} K_1 \times L_1$$

and put

$$\mathcal{F}_m = \{K_0 \times L \in \mathcal{X}_{I \times J} \times \mathcal{D} : |L| = 2^{-m}, K_0 \times L \subset B_{\tilde{I} \times J}\}, \quad (3.19c)$$

see Figure 4.

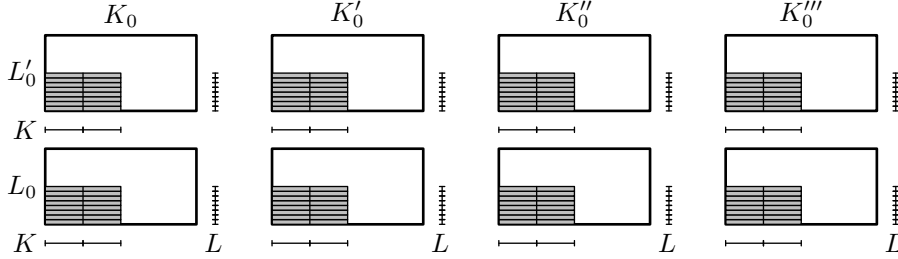


FIGURE 4. The above figure depicts an instance of  $\mathcal{F}_m$  in Case 2.b. The gray area is obtained by intersecting the gray areas of Figure 2 and 3, respectively. We form the rectangles of  $\mathcal{F}_m$  by leaving intact the intervals of the  $x$ -coordinate arising in Figure 2 and using a high frequency cover of intervals contained in the  $y$ -coordinate of Figure 3. This construction leads directly to Capon's local product structure.

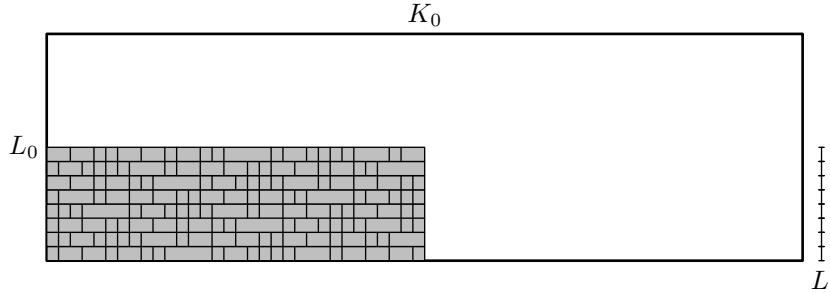


FIGURE 5. A bad cover of  $K_0 \times L_0$ : These fragmented shaded rectangles cover the same subset of  $K_0 \times L_0$  as  $\{R \in \mathcal{F}_m : R \subset K_0 \times L_0\}$ , depicted in the lower left corner of Figure 4. The  $x$ -components of the rectangles in Figure 5 do not coincide with the ones in Figure 2. The construction of block basis based on covers in Figure 5 would *not* result in block basis with Capon's local product structure.

In any of the above cases (3.18) (3.19) we define the following function. For any choice of signs  $\varepsilon_{K \times L} \in \{-1, +1\}$ ,  $K \times L \in \mathcal{F}_m$  put

$$f_m^{(\varepsilon)} = \sum_{K \times L \in \mathcal{F}_m} \varepsilon_{K \times L} h_{K \times L}, \quad (3.20)$$

recall (3.13) and define

$$R_m^{(\varepsilon)} = \sum_{K \times L \in \mathcal{F}_m} \varepsilon_{K \times L} r_{K \times L}. \quad (3.21)$$

For all choices of signs  $\varepsilon$  we have that  $f_m^{(\varepsilon)} \rightarrow 0$  weakly in  $H^p(H^q)$  and in its dual  $(H^p(H^q))^*$  as  $m \rightarrow \infty$ . Consequently, we obtain

$$\sum_{j=1}^{i-1} |\langle T b_j^{(\varepsilon)}, f_m^{(\varepsilon)} \rangle| + |\langle f_m^{(\varepsilon)}, T^* b_j^{(\varepsilon)} \rangle| \leq \eta 4^{-i} \|f_m^{(\varepsilon)}\|_2^2 \quad (3.22)$$

for all choices of signs  $\varepsilon$  and sufficiently large  $m$ . We want to point out that  $\|f_m^{(\varepsilon)}\|_2^2 = |I \times J|$ .

Given  $m$  we define the random variable

$$X(\varepsilon) = \langle f_m^{(\varepsilon)}, R_m^{(\varepsilon)} \rangle, \quad (3.23)$$

for all choices of signs  $\varepsilon_{K \times L}$ ,  $K \times L \in \mathcal{F}_m$  and denote the averaging over all those choices of signs by  $\mathbb{E}_\varepsilon$ . Now we will choose  $\varepsilon$  such that  $|X(\varepsilon)|$  is small. To accomplish this task, we will estimate  $\mathbb{E}_\varepsilon X^2$  and use Chebyshev's inequality. Using  $\langle h_{K \times L}, r_{K \times L} \rangle = 0$ , we have

$$X(\varepsilon) = \sum \varepsilon_{K_0 \times L_0} \varepsilon_{K_1 \times L_1} \langle h_{K_0 \times L_0}, r_{K_1 \times L_1} \rangle, \quad (3.24)$$

where the sum is taken over all  $K_0 \times L_0, K_1 \times L_1 \in \mathcal{F}_m$  with  $K_0 \times L_0 \neq K_1 \times L_1$ . Thus,

$$\mathbb{E}_\varepsilon X = 0. \quad (3.25)$$

From now on we shall abbreviate  $K_j \times L_j$  by  $R_j$ . From (3.24) we obtain

$$X^2(\varepsilon) = \sum \varepsilon_{R_0} \varepsilon_{R_1} \varepsilon_{R_2} \varepsilon_{R_3} \langle h_{R_0}, r_{R_1} \rangle \langle h_{R_2}, r_{R_3} \rangle, \quad (3.26)$$

where the sum is taken over all  $R_j \in \mathcal{F}_m$ ,  $0 \leq j \leq 3$  with  $R_0 \neq R_1$  and  $R_2 \neq R_3$ . Note that under the above restrictions, the following two conditions are equivalent:

- (i)  $\varepsilon_{R_0} \varepsilon_{R_1} \varepsilon_{R_2} \varepsilon_{R_3} \neq 0$ ,
- (ii)  $R_0 = R_2, R_1 = R_3$  or  $R_0 = R_3, R_1 = R_2$ .

Hence, (3.26) implies

$$\mathbb{E}_\varepsilon X^2 = \sum \langle h_{R_0}, r_{R_1} \rangle^2 + \langle h_{R_0}, r_{R_1} \rangle \langle h_{R_1}, r_{R_0} \rangle = A + B. \quad (3.27)$$

where the sum is taken over all  $R_0, R_1 \in \mathcal{F}_m$  with  $R_0 \neq R_1$ . Before we begin estimating (3.27), we record the following simple estimates:

$$\|r_{K \times L}\|_{H^p(H^q)} \leq 2\|T\| |K|^{1/p} |L|^{1/q}, \quad (3.28a)$$

$$\left\| \sum c_{K \times L} h_{K \times L} \right\|_{H^u(H^v)} \leq \sup |c_{K \times L}| |I|^{1/u} |J|^{1/v}, \quad 1 \leq u, v \leq \infty, \quad (3.28b)$$

$$\left\| \sum c_{K \times L} r_{K \times L} \right\|_{H^p(H^q)} \leq 2\|T\| \sup |c_{K \times L}| |I|^{1/p} |J|^{1/q}, \quad (3.28c)$$

where the sums and suprema are taken over all  $K \times L \in \mathcal{F}_m$ . For estimate (3.28c) we used (3.13), (3.14) and (3.28b). First, we rewrite  $A$  defined in (3.27) as

$$A = \sum_{R_0 \in \mathcal{F}_m} \left| \langle h_{R_0}, \sum_{R_1 \in \mathcal{F}_m} \langle h_{R_0}, r_{R_1} \rangle r_{R_1} \rangle \right|. \quad (3.29)$$

Observe that per construction  $|K_0| = |K_1|$  and  $|L_0| = |L_1|$  for all  $R_0, R_1 \in \mathcal{F}_m$ , thus we obtain from (3.28)

$$A \leq 4\|T\|^2 |I|^{1+1/p} |J|^{1+1/q} |K|^{1/p'} |L|^{1/q'}. \quad (3.30)$$

Second, by rewriting  $A$  as

$$A = \sum_{R_1 \in \mathcal{F}_m} \left| \left\langle \sum_{R_0 \in \mathcal{F}_m} \langle h_{R_0}, r_{R_1} \rangle h_{R_0}, r_{R_1} \right\rangle \right|, \quad (3.31)$$

we obtain from (3.28)

$$A \leq 4\|T\|^2 |I|^{1+1/p'} |J|^{1+1/q'} |K|^{1/p} |L|^{1/q}. \quad (3.32)$$

Combining (3.30) with (3.32) yields

$$A \leq 4\|T\|^2 |I|^{3/2} |J|^{3/2} |K|^{1/2} |L|^{1/2}, \quad (3.33)$$

where  $K \times L \in \mathcal{F}_m$ . Similarly, by rewriting  $B$  in (3.27), we also obtain the estimate

$$B \leq 4\|T\|^2 |I|^{3/2} |J|^{3/2} |K|^{1/2} |L|^{1/2}, \quad (3.34)$$

where  $K \times L \in \mathcal{F}_m$ . Plugging (3.33) and (3.34) into (3.27) we obtain

$$\mathbb{E}_\varepsilon X^2 \leq C |K|^{1/2} |L|^{1/2}, \quad (3.35)$$

where  $K \times L$  is an arbitrary element in  $\mathcal{F}_m$ , and the constant  $C > 0$  depends only on  $T$  and  $I \times J$ . Chebyshev's inequality, (3.23), (3.25) and (3.35), as well as considering the fact that  $\|f_m^{(\varepsilon)}\|_2^2 = |I||J|$  yield

$$\mathbb{P}(|\langle f_m^{(\varepsilon)}, R_m^{(\varepsilon)} \rangle| \geq \eta \|f_m^{(\varepsilon)}\|_2^2) \leq \frac{C}{\eta} |K|^{1/2} |L|^{1/2}, \quad (3.36)$$

for all  $K \times L \in \mathcal{F}_m$ , and the constant  $C > 0$  depends only on  $T$  and  $I \times J$ . The probability measure  $\mathbb{P}$  is the normalized counting measure on the space of signs given by  $\varepsilon_{K \times L}$ ,  $K \times L \in \mathcal{F}_m$ . Since  $|K||L| \rightarrow 0$  as  $m \rightarrow \infty$ , see (3.18) and (3.19), the concentration estimate (3.36) allows us to find an integer  $m$  and a suitable choice of signs  $\varepsilon_{K \times L}$ ,  $K \times L \in \mathcal{F}_m$  such that

$$|\langle f_m^{(\varepsilon)}, R_m^{(\varepsilon)} \rangle| \leq \eta \|f_m^{(\varepsilon)}\|_2^2. \quad (3.37)$$

From (3.13), (3.20) and (3.21) we obtain

$$\langle T f_m^{(\varepsilon)}, f_m^{(\varepsilon)} \rangle = \sum_{K \times L \in \mathcal{F}_m} \alpha_{K \times L} |K \times L| + \langle f_m^{(\varepsilon)}, R_m^{(\varepsilon)} \rangle.$$

Estimating the latter identity using (3.14) and (3.37) yields

$$\langle T f_m^{(\varepsilon)}, f_m^{(\varepsilon)} \rangle \geq (\delta - \eta) \|f_m^{(\varepsilon)}\|_2^2. \quad (3.38)$$

We conclude the inductive construction step by defining

$$\mathcal{B}_i = \mathcal{F}_m \quad \text{and} \quad b_i^{(\varepsilon)} = f_m^{(\varepsilon)}, \quad (3.39)$$

where we choose  $m$  and  $\varepsilon$  according to (3.22) and (3.37).

### Essential properties of our inductive construction.

*Mixed-norm estimates for  $b_{I \times J}^{(\varepsilon)}$ .*

Given  $1 \leq u, v \leq \infty$  and  $I \times J \in \mathcal{B}$ , we record the following estimates.

$$\|b_{I \times J}^{(\varepsilon)}\|_2^2 = |I \times J| = \|h_{I \times J}\|_2^2. \quad (3.40a)$$

$$\|b_{I \times J}^{(\varepsilon)}\|_{H^u(H^v)} = |I|^{1/u} |J|^{1/v} = \|h_{I \times J}\|_{H^u(H^v)}. \quad (3.40b)$$

*The block basis  $\{b_i^{(\varepsilon)}\}$  almost-diagonalizes  $T$ .*

We show that we have

$$T b_i^{(\varepsilon)} = \frac{|\langle T b_i^{(\varepsilon)}, b_i^{(\varepsilon)} \rangle|}{\|b_i^{(\varepsilon)}\|_2^2} b_i^{(\varepsilon)} + \text{tiny error}.$$

Note estimate (3.17c) for the diagonal. We now calculate the size of the error terms.

We claim that for all  $i$  we have

$$\sum_{j: j \neq i} j |\langle T b_j^{(\varepsilon)}, b_i^{(\varepsilon)} \rangle| \leq \eta 2^{-i} \|b_i^{(\varepsilon)}\|_2^2. \quad (3.41)$$

Note that (3.17b) implies

$$\sum_{j=1}^{i-1} j |\langle T b_j^{(\varepsilon)}, b_i^{(\varepsilon)} \rangle| \leq \eta i 4^{-i} \|b_i^{(\varepsilon)}\|_2^2, \quad (3.42a)$$

$$\sum_{j=i+1}^{\infty} j |\langle T b_j^{(\varepsilon)}, b_i^{(\varepsilon)} \rangle| \leq \eta \sum_{j=i+1}^{\infty} j 4^{-j} \|b_j^{(\varepsilon)}\|_2^2. \quad (3.42b)$$

Since  $\|b_j^{(\varepsilon)}\|_2^2 < \frac{1+\sqrt{j}}{4} \|b_i^{(\varepsilon)}\|_2^2$  by (3.11), we obtain (3.41) from (3.42).

$\mathcal{B}_{I \times J}$  satisfies Capon's local product condition.

The collection  $\{\mathcal{B}_{I \times J} : I \times J \in \mathcal{R}\}$  satisfies Capon's local product condition. We write

$$\mathcal{B}_{I \times J} = \{K \times L : K \in \mathcal{X}_I, L \in \mathcal{Y}_J(K)\},$$

with  $\mathcal{X}_I, \mathcal{Y}_J(K) \subset \mathcal{D}$ . First we note that since the collections  $\mathcal{X}_I, I \in \mathcal{D}$  are the result of a Gamlen-Gaudet process, the conditions (X1)–(X5) hold with  $C_X = 1$ . Second, with  $K \in \mathcal{X}$  fixed, the collections  $\mathcal{Y}_J(K), J \in \mathcal{D}$  are the result of a Gamlen-Gaudet process, therefore the conditions (Y1), (Y2) and (Y5) hold. Third, observe that for each  $x$  we have  $Y_{J,x} = Y_J^x = Y_J(K)$  for all  $K \ni x$ , therefore the conditions (Y3) and (Y4) are satisfied. The constant  $C_Y$  is given by  $C_Y = 1$ .

### Putting it together.

Since  $\mathcal{B}_{I \times J}$  satisfies Capon's local product condition, we obtain from Theorem 3.2 the following two results. First, let  $Y = \text{span}\{b_i^{(\varepsilon)}\} \subset H^p(H^q)$  and define  $B : H^p(H^q) \rightarrow Y$  by  $Bh_i = b_i^{(\varepsilon)}$ , then

$$\begin{array}{ccc} H^p(H^q) & \xrightarrow{I} & H^p(H^q) \\ B \downarrow & & \uparrow B^{-1} \\ Y & \xrightarrow{I} & Y \end{array} \quad \|B\| \|B^{-1}\| \leq C, \quad (3.43)$$

for some universal constant  $C > 0$ . Second, the orthogonal projection  $Q : H^p(H^q) \rightarrow H^p(H^q)$  defined in (3.5) satisfies the estimate

$$\|Q : H^p(H^q) \rightarrow H^p(H^q)\| \leq C, \quad (3.44)$$

for some universal constant  $C > 0$ .

Now, define  $U : H^p(H^q) \rightarrow Y$  by

$$U(f) = \sum_{i=1}^{\infty} \frac{\langle f, b_i^{(\varepsilon)} \rangle}{\langle T b_i^{(\varepsilon)}, b_i^{(\varepsilon)} \rangle} b_i^{(\varepsilon)}, \quad (3.45)$$

and note that (3.17c) and (3.44) imply

$$\|U : H^p(H^q) \rightarrow Y\|_{H^p(H^q)} \leq \frac{C}{\delta - \eta}. \quad (3.46)$$

Observe that for all  $g = \sum_{i=1}^{\infty} \lambda_i b_i^{(\varepsilon)} \in Y$  we have the identity

$$UTg - g = \sum_{i=1}^{\infty} \left( \sum_{j:j \neq i} \lambda_j \langle T b_j^{(\varepsilon)}, b_i^{(\varepsilon)} \rangle \right) \langle T b_i^{(\varepsilon)}, b_i^{(\varepsilon)} \rangle^{-1} b_i^{(\varepsilon)}.$$

We obtain from (3.40) and (3.11) that  $|\lambda_j| \leq 4j \|g\|_{H^p(H^q)}$ , so using (3.17c) and (3.41) yields

$$\|UTg - g\|_{H^p(H^q)} \leq \frac{\eta}{\delta - \eta} \|g\|_{H^p(H^q)}.$$

Finally, let  $J : Y \rightarrow H^p(H^q)$  denote the operator given by  $Jy = y$ . If we choose  $\eta > 0$  small enough (e.g.  $\eta = \delta/9$ ) and define the operator  $V : H^p(H^q) \rightarrow Y$  by  $(UTJ)^{-1}U$ , then

$$\begin{array}{ccc}
 Y & \xrightarrow{I} & Y \\
 \downarrow J & \searrow UTJ & \nearrow (UTJ)^{-1} \\
 & Y & \\
 & \nwarrow U & \uparrow V \\
 H^p(H^q) & \xrightarrow{T} & H^p(H^q)
 \end{array} \quad \|J\| \|V\| \leq C/\delta, \quad (3.47)$$

for some universal constant  $C > 0$ .

Merging the commutative diagram (3.43) with (3.47) concludes the proof.  $\square$

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